

GLOBAL ATTRACTORS FOR THE PLATE EQUATION WITH NONLOCAL NONLINEARITY IN UNBOUNDED DOMAINS

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ABSTRACT. We consider the initial value problem for the semilinear plate equation with nonlocal nonlinearity. We prove the existence of global attractor and then establish the regularity and finite dimensionality of this attractor.

1. INTRODUCTION

The main aim of this paper is to study the long time dynamics (in terms of attractors) of the plate equation

$$u_{tt} + \Delta^2 u + \alpha(x)u_t + \lambda u - f(\|\nabla u(t)\|_{L^2(\mathbb{R}^n)})\Delta u = h(x), \quad (t, x) \in (0, \infty) \times \mathbb{R}^n, \quad (1.1)$$

with initial data

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbb{R}^n, \quad (1.2)$$

where $\lambda > 0$, $h \in L^2(\mathbb{R}^n)$ and the functions $\alpha(\cdot)$, $f(\cdot)$ satisfy the following conditions:

$$\alpha \in L^\infty(\mathbb{R}^n), \quad \alpha(\cdot) \geq \alpha_0 > 0 \quad \text{a.e. in } \mathbb{R}^n, \quad (1.3)$$

$$f \in C^1(\mathbb{R}^+), \quad f(z) \geq 0, \quad \text{for all } z \in \mathbb{R}^+. \quad (1.4)$$

By the semigroup theory, it is easy to show that under the conditions (1.3) and (1.4), for every $(u_0, u_1) \in H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, the problem (1.1)-(1.2) has a unique weak solution $u \in C([0, \infty); H^2(\mathbb{R}^n)) \cap C^1([0, \infty); L^2(\mathbb{R}^n))$, which depends continuously on the initial data and satisfies the energy equality

$$\begin{aligned} E(u(t)) + \frac{1}{2}F\left(\|\nabla u(t)\|_{L^2(\mathbb{R}^n)}^2\right) - \int_{\mathbb{R}^n} h(x)u(t, x)dx + \int_s^t \int_{\mathbb{R}^n} \alpha(x)|u_t(\tau, x)|^2 dx d\tau \\ = E(u(s)) + \frac{1}{2}F\left(\|\nabla u(s)\|_{L^2(\mathbb{R}^n)}^2\right) - \int_{\mathbb{R}^n} h(x)u(s, x)dx, \quad \forall t \geq s \geq 0, \end{aligned} \quad (1.5)$$

where $F(z) = \int_0^z f(\sqrt{s})ds$ for all $z \in \mathbb{R}^+$ and $E(u(t)) = \frac{1}{2} \int_{\mathbb{R}^n} (|u_t(t, x)|^2 + |\Delta u(t, x)|^2 + \lambda |u(t, x)|^2) dx$. Moreover, if $(u_0, u_1) \in H^4(\mathbb{R}^n) \times H^2(\mathbb{R}^n)$, then u is a strong solution from the class $C([0, \infty); H^4(\mathbb{R}^n)) \cap C^1([0, \infty); H^2(\mathbb{R}^n)) \cap C^2([0, \infty); L^2(\mathbb{R}^n))$. Therefore, the problem (1.1)-(1.2) generates a strongly continuous semigroup $\{S(t)\}_{t \geq 0}$ in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ by the formula $(u(t), u_t(t)) = S(t)(u_0, u_1)$,

2000 *Mathematics Subject Classification.* 35B41, 35G20, 37L30, 74K20.

Key words and phrases. plate equation, global attractor.

where $u(t, x)$ is a weak solution of (1.1)-(1.2) with the initial data (u_0, u_1) . By (1.4) and (1.5), we have the inequality

$$E(u(t)) + \int_0^t \int_{\mathbb{R}^n} \alpha(x) |u_t(\tau, x)|^2 dx d\tau \leq c \left(\|(u_0, u_1)\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} \right), \quad \forall t \geq 0, \quad (1.6)$$

which implies the boundedness of $\{S(t)\}_{t \geq 0}$ in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, where $c : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a nondecreasing function.

The problem of investigating the asymptotic behavior of evolution equations modeling many physical phenomena has been attracting more attention over the last few decades. It is well known that the asymptotic behavior of these equations can be described by means of the attractors. The attractors for plate equations has been one of the intensively studied topic in recent years. We refer to [1-10] for attractors of plate equations with local and nonlocal nonlinearities in bounded domains. In the case of unbounded domains, there are obstacles in applying the methods given for bounded domains due to the lack of Sobolev compact embedding theorems. So as to handle these obstacles, the authors of [11-14] established the uniform tail estimates for the plate equations with local nonlinearities.

The situation becomes more difficult when the domain is unbounded and the equation includes nonlocal nonlinearity, for example $f(\|\nabla u(t)\|_{L^2(\mathbb{R}^n)})\Delta u(t)$ as in the case of equation (1.1). When $f(s) = s^2$, this nonlocal term becomes famous Berger nonlinearity (see [15]). In the unbounded domain case, the operator $\mathcal{F}(u) := f(\|\nabla u\|_{L^2(\mathbb{R}^n)})\Delta u$ which is determined by the nonlocal term mentioned above, besides being not compact, is not also weakly continuous from $H^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. So, in order to establish the asymptotic compactness which is necessary for the existence of the global attractor, we are not able to apply either the standard splitting method or the energy method devised in [16]. To overcome these difficulties, we apply compensated compactness method introduced in [17] and prove the asymptotic compactness (see Lemma 2.2) which, together with the presence of the strict Lyapunov function, leads to the existence of a global attractor. Then, by using the invariance of the global attractor and the structural property of the set of stationary points, we establish the regularity (see Theorem 3.1) and consequently, the finite dimensionality (see Theorem 4.1) of the global attractor.

Our main result is as follows:

Theorem 1.1. *Under conditions (1.3) and (1.4) the semigroup $\{S(t)\}_{t \geq 0}$ generated by the problem (1.1)-(1.2) possesses a global attractor \mathcal{A} in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and $\mathcal{A} = \mathcal{M}^u(\mathcal{N})$. Here $\mathcal{M}^u(\mathcal{N})$ is unstable manifold emanating from the set of stationary points \mathcal{N} (for definition, see [18, p.359]). Moreover, the global attractor \mathcal{A} is bounded in $H^4(\mathbb{R}^n) \times H^2(\mathbb{R}^n)$ and it has finite fractal dimension.*

Remark 1.1. We note that by using the method of this paper, one can prove the existence, regularity and finite dimensionality of the global attractor for the initial boundary value problem

$$\begin{cases} u_{tt} + \Delta^2 u + \alpha(x)u_t + \lambda u - f(\|\nabla u(t)\|_{L^2(\Omega)})\Delta u = h(x), & (t, x) \in (0, \infty) \times \Omega, \\ u(t, x) = \frac{\partial}{\partial \nu} u(t, x) = 0, & (t, x) \in (0, \infty) \times \partial\Omega, \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), & x \in \Omega, \end{cases} \quad (1.7)$$

where $\Omega \subset \mathbb{R}^n$ is an unbounded domain with smooth boundary, ν is outer unit normal vector, $\lambda > 0$, $h \in L^2(\Omega)$, the function $f(\cdot)$ satisfies the condition (1.4) and the damping coefficient $\alpha(\cdot)$ satisfies the following conditions

$$\alpha \in L^\infty(\Omega), \quad \alpha(\cdot) \geq \alpha_0 > 0, \quad \text{a.e. in } \Omega.$$

Remark 1.2. We also note that we critically use the strict positivity of $\alpha(\cdot)$ (see (1.3)) in the proof of asymptotic compactness of the semigroup $\{S(t)\}_{t \geq 0}$ (see Lemma 2.2). In the case when the function $\alpha(\cdot)$ is not strictly positive, for example if $\alpha(\cdot)$ vanishes in a set of positive measure, our method is not applicable. Another obstacle in this case is related to unique continuation of solutions which is important for the construction of a strict Lyapunov function. To the best of our knowledge the unique continuation of solutions for the equations (1.1) and (1.7)₁-(1.7)₂ is also an open question. Thus, in the case when $\alpha(\cdot)$ vanishes in a set of positive measure, the questions about long time dynamics of (1.1)-(1.2) and (1.7), in terms of attractors, are completely open.

2. EXISTENCE OF THE GLOBAL ATTRACTOR

In this section, we will show the existence of the global attractor. To this end, we first prove the following lemma.

Lemma 2.1. Let the conditions (1.3) and (1.4) hold. Also, assume that the sequence $\{v_m\}_{m=1}^\infty$ is bounded in $L^\infty(0, T; H^2(\mathbb{R}^n)) \cap W^{1,\infty}(0, T; L^2(\mathbb{R}^n))$ and the sequence $\left\{\|\nabla v_m(t)\|_{L^2(\mathbb{R}^n)}\right\}_{m=1}^\infty$ is convergent, for all $t \in [0, T]$. Then, for all $\gamma > 0$, there exists some $c_\gamma > 0$ such that

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^n} \tau \left(f\left(\|\nabla v_m(\tau)\|_{L^2(\mathbb{R}^n)}\right) \Delta v_m(\tau, x) - f\left(\|\nabla v_l(\tau)\|_{L^2(\mathbb{R}^n)}\right) \Delta v_l(\tau, x) \right) (v_{mt}(\tau, x) - v_{lt}(\tau, x)) dx d\tau \\ & \leq \gamma \int_0^t \tau E(v_m(\tau) - v_l(\tau)) d\tau + c_\gamma \int_0^t E(v_m(\tau) - v_l(\tau)) d\tau \\ & \quad + c_\gamma \int_0^t \tau E(v_m(\tau) - v_l(\tau)) \|v_{mt}(\tau)\|_{L^2(\mathbb{R}^n)}^2 d\tau + K^{m,l}(t), \end{aligned}$$

for all $t \in [0, T]$, where $K^{m,l} \in C[0, T]$ and $\limsup_{m \rightarrow \infty} \limsup_{l \rightarrow \infty} \|K^{m,l}\|_{C[0, T]} = 0$.

Proof. Firstly, we have

$$\int_0^t \int_{\mathbb{R}^n} \tau \left(f\left(\|\nabla v_m(\tau)\|_{L^2(\mathbb{R}^n)}\right) \Delta v_m(\tau, x) - f\left(\|\nabla v_l(\tau)\|_{L^2(\mathbb{R}^n)}\right) \Delta v_l(\tau, x) \right) (v_{mt}(\tau, x) - v_{lt}(\tau, x)) dx d\tau$$

$$= - \int_0^t \tau f \left(\|\nabla v_m(\tau)\|_{L^2(\mathbb{R}^n)} \right) \frac{d}{d\tau} \left(\|\nabla v_m(\tau) - \nabla v_l(\tau)\|_{L^2(\mathbb{R}^n)}^2 \right) d\tau + K^{m,l}(t), \quad (2.1)$$

where

$$K^{m,l}(t) := \int_0^t \int_{\mathbb{R}^n} \tau \left(f \left(\|\nabla v_m(\tau)\|_{L^2(\mathbb{R}^n)} \right) - f \left(\|\nabla v_l(\tau)\|_{L^2(\mathbb{R}^n)} \right) \right) \Delta v_l(\tau, x) (v_{mt}(\tau, x) - v_{lt}(\tau, x)) dx d\tau.$$

By the conditions of the lemma, we obtain

$$\limsup_{m \rightarrow \infty} \limsup_{l \rightarrow \infty} \|K^{m,l}\|_{C[0,T]} = 0. \quad (2.2)$$

Now, let us estimate the first term on the right side of (2.1). For any $\varepsilon > 0$, by integration by parts, we have

$$\begin{aligned} & \int_0^t \tau f \left(\|\nabla v_m(\tau)\|_{L^2(\mathbb{R}^n)} \right) \frac{d}{d\tau} \left(\|\nabla v_m(\tau) - \nabla v_l(\tau)\|_{L^2(\mathbb{R}^n)}^2 \right) d\tau \\ &= \int_0^t \tau \left(f \left(\|\nabla v_m(\tau)\|_{L^2(\mathbb{R}^n)} \right) - f(\varepsilon) \right) \frac{d}{d\tau} \left(\|\nabla v_m(\tau) - \nabla v_l(\tau)\|_{L^2(\mathbb{R}^n)}^2 \right) d\tau \\ & \quad + f(\varepsilon) \int_0^t \tau \frac{d}{d\tau} \left(\|\nabla v_m(\tau) - \nabla v_l(\tau)\|_{L^2(\mathbb{R}^n)}^2 \right) d\tau \\ &= \int_{A_{1,\varepsilon}^m(t)} \tau \left(f \left(\|\nabla v_m(\tau)\|_{L^2(\mathbb{R}^n)} \right) - f(\varepsilon) \right) \frac{d}{d\tau} \left(\|\nabla v_m(\tau) - \nabla v_l(\tau)\|_{L^2(\mathbb{R}^n)}^2 \right) d\tau \\ & \quad + \int_{A_{2,\varepsilon}^m(t)} \tau \left(f \left(\|\nabla v_m(\tau)\|_{L^2(\mathbb{R}^n)} \right) - f(\varepsilon) \right) \frac{d}{d\tau} \left(\|\nabla v_m(\tau) - \nabla v_l(\tau)\|_{L^2(\mathbb{R}^n)}^2 \right) d\tau \\ & \quad + t f(\varepsilon) \|\nabla v_m(t) - \nabla v_l(t)\|_{L^2(\mathbb{R}^n)}^2 - f(\varepsilon) \int_0^t \|\nabla v_m(\tau) - \nabla v_l(\tau)\|_{L^2(\mathbb{R}^n)}^2 d\tau, \end{aligned} \quad (2.3)$$

where

$$\begin{aligned} A_{1,\varepsilon}^m(t) &:= \left\{ \tau \in (0, t) : \|\nabla v_m(\tau)\|_{L^2(\mathbb{R}^n)} \leq \varepsilon \right\}, \\ A_{2,\varepsilon}^m(t) &:= \left\{ \tau \in (0, t) : \|\nabla v_m(\tau)\|_{L^2(\mathbb{R}^n)} > \varepsilon \right\}. \end{aligned}$$

Let us estimate the term

$$\int_{A_{2,\varepsilon}^m(t)} \tau \left(f \left(\|\nabla v_m(\tau)\|_{L^2(\mathbb{R}^n)} \right) - f(\varepsilon) \right) \frac{d}{d\tau} \left(\|\nabla v_m(\tau) - \nabla v_l(\tau)\|_{L^2(\mathbb{R}^n)}^2 \right) d\tau.$$

It is enough to consider the case $A_{2,\varepsilon}^m(t)$ is nonempty. Since, thanks to $v_m \in C([0, T]; H^1(\mathbb{R}^n))$, the set $A_{2,\varepsilon}^m(t)$ is open, it can be shown that it is a countable union of disjoint open intervals $\{(t_k, \tilde{t}_k)\}_{k=1}^\infty$ (see, for example [19, p. 39]). Then

$$\int_{A_{2,\varepsilon}^m(t)} \tau \left(f \left(\|\nabla v_m(\tau)\|_{L^2(\mathbb{R}^n)} \right) - f(\varepsilon) \right) \frac{d}{d\tau} \left(\|\nabla v_m(\tau) - \nabla v_l(\tau)\|_{L^2(\mathbb{R}^n)}^2 \right) d\tau$$

$$= \sum_{k=0}^{\infty} \int_{t_k}^{\tilde{t}_k} \tau \left(f \left(\|\nabla v_m(\tau)\|_{L^2(\mathbb{R}^n)} \right) - f(\varepsilon) \right) \frac{d}{d\tau} \left(\|\nabla v_m(\tau) - \nabla v_l(\tau)\|_{L^2(\mathbb{R}^n)}^2 \right) d\tau.$$

If $\|\nabla v_m(t)\|_{L^2(\mathbb{R}^n)} \leq \varepsilon$, then by continuity of f , we have

$$f \left(\|\nabla v_m(t_k)\|_{L^2(\mathbb{R}^n)} \right) = f(\varepsilon), \quad k = 1, 2, \dots,$$

$$f \left(\|\nabla v_m(\tilde{t}_k)\|_{L^2(\mathbb{R}^n)} \right) = f(\varepsilon), \quad k = 1, 2, \dots.$$

Hence, by integration by parts, we obtain

$$\begin{aligned} & \int_{A_{2,\varepsilon}^m(t)} \tau \left(f \left(\|\nabla v_m(\tau)\|_{L^2(\mathbb{R}^n)} \right) - f(\varepsilon) \right) \frac{d}{d\tau} \left(\|\nabla v_m(\tau) - \nabla v_l(\tau)\|_{L^2(\mathbb{R}^n)}^2 \right) d\tau \\ &= \sum_{k=1}^{\infty} \tilde{t}_k \left(f \left(\|\nabla v_m(\tilde{t}_k)\|_{L^2(\mathbb{R}^n)} \right) - f(\varepsilon) \right) \|\nabla v_m(\tilde{t}_k) - \nabla v_l(\tilde{t}_k)\|_{L^2(\mathbb{R}^n)}^2 \\ &\quad - \sum_{k=1}^{\infty} t_k \left(f \left(\|\nabla v_m(t_k)\|_{L^2(\mathbb{R}^n)} \right) - f(\varepsilon) \right) \|\nabla v_m(t_k) - \nabla v_l(t_k)\|_{L^2(\mathbb{R}^n)}^2 \\ &\quad - \sum_{k=1}^{\infty} \int_{t_k}^{\tilde{t}_k} \left(f \left(\|\nabla v_m(\tau)\|_{L^2(\mathbb{R}^n)} \right) - f(\varepsilon) \right) \|\nabla v_m(\tau) - \nabla v_l(\tau)\|_{L^2(\mathbb{R}^n)}^2 d\tau \\ &+ \sum_{k=1}^{\infty} \int_{t_k}^{\tilde{t}_k} \tau \frac{f' \left(\|\nabla v_m(\tau)\|_{L^2(\mathbb{R}^n)} \right)}{\|\nabla v_m(\tau)\|_{L^2(\mathbb{R}^n)}} \langle \Delta v_m(\tau), v_{mt}(\tau) \rangle_{L^2(\mathbb{R}^n)} \|\nabla v_m(\tau) - \nabla v_l(\tau)\|_{L^2(\mathbb{R}^n)}^2 d\tau \\ &= - \int_{A_{2,\varepsilon}^m(t)} \left(f \left(\|\nabla v_m(\tau)\|_{L^2(\mathbb{R}^n)} \right) - f(\varepsilon) \right) \|\nabla v_m(\tau) - \nabla v_l(\tau)\|_{L^2(\mathbb{R}^n)}^2 d\tau \\ &+ \int_{A_{2,\varepsilon}^m(t)} \tau \frac{f' \left(\|\nabla v_m(\tau)\|_{L^2(\mathbb{R}^n)} \right)}{\|\nabla v_m(\tau)\|_{L^2(\mathbb{R}^n)}} \langle \Delta v_m(\tau), v_{mt}(\tau) \rangle_{L^2(\mathbb{R}^n)} \|\nabla v_m(\tau) - \nabla v_l(\tau)\|_{L^2(\mathbb{R}^n)}^2 d\tau, \quad (2.4) \end{aligned}$$

where $\langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^n)}$ is an inner product in $L^2(\mathbb{R}^n)$. If $\|\nabla v_m(t)\|_{L^2(\mathbb{R}^n)} > \varepsilon$, then the maximal element of $\{\tilde{t}_k : k = 1, 2, \dots\}$ is equal to t and consequently, we have

$$\begin{aligned} & \int_{A_{2,\varepsilon}^m(t)} \tau \left(f \left(\|\nabla v_m(\tau)\|_{L^2(\mathbb{R}^n)} \right) - f(\varepsilon) \right) \frac{d}{d\tau} \left(\|\nabla v_m(\tau) - \nabla v_l(\tau)\|_{L^2(\mathbb{R}^n)}^2 \right) d\tau \\ &= \sum_{k=1}^{\infty} \tilde{t}_k \left(f \left(\|\nabla v_m(\tilde{t}_k)\|_{L^2(\mathbb{R}^n)} \right) - f(\varepsilon) \right) \|\nabla v_m(\tilde{t}_k) - \nabla v_l(\tilde{t}_k)\|_{L^2(\mathbb{R}^n)}^2 \\ &\quad - \sum_{k=1}^{\infty} t_k \left(f \left(\|\nabla v_m(t_k)\|_{L^2(\mathbb{R}^n)} \right) - f(\varepsilon) \right) \|\nabla v_m(t_k) - \nabla v_l(t_k)\|_{L^2(\mathbb{R}^n)}^2 \\ &\quad - \sum_{k=1}^{\infty} \int_{t_k}^{\tilde{t}_k} \left(f \left(\|\nabla v_m(\tau)\|_{L^2(\mathbb{R}^n)} \right) - f(\varepsilon) \right) \|\nabla v_m(\tau) - \nabla v_l(\tau)\|_{L^2(\mathbb{R}^n)}^2 d\tau \\ &+ \sum_{k=1}^{\infty} \int_{t_k}^{\tilde{t}_k} \tau \frac{f' \left(\|\nabla v_m(\tau)\|_{L^2(\mathbb{R}^n)} \right)}{\|\nabla v_m(\tau)\|_{L^2(\mathbb{R}^n)}} \langle \Delta v_m(\tau), v_{mt}(\tau) \rangle_{L^2(\mathbb{R}^n)} \|\nabla v_m(\tau) - \nabla v_l(\tau)\|_{L^2(\mathbb{R}^n)}^2 d\tau \end{aligned}$$

$$\begin{aligned}
&= t \left(f \left(\|\nabla v_m(t)\|_{L^2(\mathbb{R}^n)} \right) - f(\varepsilon) \right) \|\nabla v_m(t) - \nabla v_l(t)\|_{L^2(\mathbb{R}^n)}^2 \\
&- \int_{A_{2,\varepsilon}^m(t)} \left(f \left(\|\nabla v_m(\tau)\|_{L^2(\mathbb{R}^n)} \right) - f(\varepsilon) \right) \|\nabla v_m(\tau) - \nabla v_l(\tau)\|_{L^2(\mathbb{R}^n)}^2 d\tau \\
&+ \int_{A_{2,\varepsilon}^m(t)} \tau \frac{f' \left(\|\nabla v_m(\tau)\|_{L^2(\mathbb{R}^n)} \right)}{\|\nabla v_m(\tau)\|_{L^2(\mathbb{R}^n)}} \langle \Delta v_m(\tau), v_{mt}(\tau) \rangle_{L^2(\mathbb{R}^n)} \|\nabla v_m(\tau) - \nabla v_l(\tau)\|_{L^2(\mathbb{R}^n)}^2 d\tau. \quad (2.5)
\end{aligned}$$

Hence, by using (1.4), (2.4) and (2.5) in (2.3), we find

$$\begin{aligned}
&- \int_0^t \tau f \left(\|\nabla v_m(\tau)\|_{L^2(\mathbb{R}^n)} \right) \frac{d}{d\tau} \left(\|\nabla v_m(\tau) - \nabla v_l(\tau)\|_{L^2(\mathbb{R}^n)}^2 \right) d\tau \\
&\leq f(\varepsilon) \int_0^t \|\nabla v_m(\tau) - \nabla v_l(\tau)\|_{L^2(\mathbb{R}^n)}^2 d\tau + \widehat{c}_1 \int_0^t \|\nabla v_m(\tau) - \nabla v_l(\tau)\|_{L^2(\mathbb{R}^n)}^2 d\tau \\
&\quad + 2 \max_{s_1, s_2 \in [0, \varepsilon]} |f(s_1) - f(s_2)| \int_0^t \tau E(v_m(\tau) - v_l(\tau)) d\tau \\
&\quad + \frac{\widehat{c}_1}{\varepsilon} \int_0^t \tau E(v_m(\tau) - v_l(\tau)) \|v_{mt}(\tau)\|_{L^2(\mathbb{R}^n)} d\tau \\
&\leq (f(\varepsilon) + \widehat{c}_1) \int_0^t \|\nabla v_m(\tau) - \nabla v_l(\tau)\|_{L^2(\mathbb{R}^n)}^2 d\tau \\
&\quad + \left(2 \max_{s_1, s_2 \in [0, \varepsilon]} |f(s_1) - f(s_2)| + \widehat{c}_1 \varepsilon \right) \int_0^t \tau E(v_m(\tau) - v_l(\tau)) d\tau \\
&\quad + \frac{\widehat{c}_1}{\varepsilon^3} \int_0^t \tau E(v_m(\tau) - v_l(\tau)) \|v_{mt}(\tau)\|_{L^2(\mathbb{R}^n)}^2 d\tau, \quad \forall t \in [0, T]. \quad (2.6)
\end{aligned}$$

Thus, setting $\gamma = 2 \max_{s_1, s_2 \in [0, \varepsilon]} |f(s_1) - f(s_2)| + \widehat{c}_1 \varepsilon$ and $c_\gamma = \max \left\{ \frac{\widehat{c}_1}{\varepsilon^3}, f(\varepsilon) + \widehat{c}_1 \right\}$, by (2.1), (2.2) and (2.6), we get the claim of the lemma. \square

Now, let us prove the asymptotic compactness of $\{S(t)\}_{t \geq 0}$ in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

Lemma 2.2. *Assume that the conditions (1.3)-(1.4) hold and B is a bounded subset of $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. Then for every sequence of the form $\{S(t_k)\varphi_k\}_{k=1}^\infty$, where $\{\varphi_k\}_{k=1}^\infty \subset B$, $t_k \rightarrow \infty$, has a convergent subsequence in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.*

Proof. Since $\{\varphi_k\}_{k=1}^\infty$ is bounded in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, by (1.3), (1.4) and (1.6) it follows that the sequence $\{S(\cdot)\varphi_k\}_{k=1}^\infty$ is bounded in $C_b(0, \infty; H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n))$, where $C_b(0, \infty; H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n))$ is the space of continuously bounded functions from $[0, \infty)$ to $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. Then for any $T \geq 0$ there exists a subsequence $\{k_m\}_{m=1}^\infty$ such that $t_{k_m} \geq T$, and

$$\|\nabla v_m(t)\|_{L^2(\mathbb{R}^n)}^2 \rightarrow q(t) \text{ weakly star in } W^{1,\infty}(0, \infty), \quad (2.7)$$

for some $q \in W^{1,\infty}(0, \infty)$, where $(v_m(t), v_{mt}(t)) = S(t + t_{k_m} - T)\varphi_{k_m}$.

Taking into account (1.4) and (1.6), we find

$$\int_0^T \|v_{mt}(t)\|_{L^2(\mathbb{R}^n)}^2 dt \leq c_1, \quad \forall T \geq 0. \quad (2.8)$$

By (1.1)₁, we have

$$\begin{aligned} v_{mtt}(t, x) - v_{ltt}(t, x) + \Delta^2(v_m(t, x) - v_l(t, x)) + \alpha(x)(v_{mt}(t, x) - v_{lt}(t, x)) + \lambda(v_m(t, x) - v_l(t, x)) \\ - f(\|\nabla v_m(t)\|_{L^2(\mathbb{R}^n)})\Delta v_m(t, x) + f(\|\nabla v_l(t)\|_{L^2(\mathbb{R}^n)})\Delta v_l(t, x) = 0. \end{aligned} \quad (2.9)$$

Multiplying (2.9) by $(v_m - v_l)$ and integrating over $(0, T) \times \mathbb{R}^n$, we get

$$\begin{aligned} \int_0^T \|\Delta(v_m(t) - v_l(t))\|_{L^2(\mathbb{R}^n)}^2 dt + \lambda \int_0^T \|v_m(t) - v_l(t)\|_{L^2(\mathbb{R}^n)}^2 dt \\ + \int_0^T f(\|\nabla v_m(t)\|_{L^2(\mathbb{R}^n)}) \|\nabla v_m(t) - \nabla v_l(t)\|_{L^2(\mathbb{R}^n)}^2 dt \leq c_2 + c_2 \int_0^T \|v_{mt}(t) - v_{lt}(t)\|_{L^2(\mathbb{R}^n)}^2 dt \\ + \int_0^T \left| f(\|\nabla v_m(t)\|_{L^2(\mathbb{R}^n)}) - f(\|\nabla v_l(t)\|_{L^2(\mathbb{R}^n)}) \right| \|\nabla v_l(t, x)\|_{L^2(\mathbb{R}^n)} \|\nabla v_m(t) - \nabla v_l(t)\|_{L^2(\mathbb{R}^n)} dt. \end{aligned}$$

Taking into account (1.4), (2.7) and (2.8) in the last inequality and passing to the limit, we obtain

$$\limsup_{m \rightarrow \infty} \limsup_{l \rightarrow \infty} \int_0^T E(v_m(t) - v_l(t)) dt \leq c_3, \quad \forall T \geq 0. \quad (2.10)$$

Multiplying (2.9) by $t(v_{mt} - v_{lt})$, integrating over $(0, T) \times \mathbb{R}^n$ and using integration by parts, by (1.3), we find

$$\begin{aligned} T E(v_m(T) - v_l(T)) + \alpha_0 \int_0^T t \|v_{mt}(T) - v_{lt}(T)\|_{L^2(\mathbb{R}^n)}^2 dt \leq \int_0^T E(v_m(t) - v_l(t)) dt \\ + \int_0^T \int_{\mathbb{R}^n} t \left(f(\|\nabla v_m(t)\|_{L^2(\mathbb{R}^n)})\Delta v_m(t, x) - f(\|\nabla v_l(t)\|_{L^2(\mathbb{R}^n)})\Delta v_l(t, x) \right) (v_{mt}(t, x) - v_{lt}(t, x)) dx dt, \end{aligned}$$

which, together with Lemma 2.1, gives

$$\begin{aligned} T E(v_m(T) - v_l(T)) + \alpha_0 \int_0^T t \|v_{mt}(T) - v_{lt}(T)\|_{L^2(\mathbb{R}^n)}^2 dt \\ \leq \int_0^T E(v_m(t) - v_l(t)) dt + \gamma \int_0^T t E(v_m(t) - v_l(t)) dt + c_\gamma \int_0^T E(v_m(t) - v_l(t)) dt \\ + c_\gamma \int_0^T t E(v_m(t) - v_l(t)) \|v_{mt}(t)\|_{L^2(\mathbb{R}^n)}^2 dt + K^{m,l}(T), \end{aligned} \quad (2.11)$$

for every $\gamma > 0$. Multiplying (2.9) by $\varepsilon t (v_m - v_l)$ and integrating over $(0, T) \times \mathbb{R}^n$, we get

$$\begin{aligned} & \varepsilon \int_0^T t \|\Delta (v_m(t) - v_l(t))\|_{L^2(\mathbb{R}^n)}^2 dt + \varepsilon \lambda \int_0^T t \|v_m(t) - v_l(t)\|_{L^2(\mathbb{R}^n)}^2 dt \\ & \leq \varepsilon c_3 T E(v_m(T) - v_l(T)) + \varepsilon \int_0^T t \|v_{mt}(t) - v_{lt}(t)\|_{L^2(\mathbb{R}^n)}^2 dt \\ & \quad + \varepsilon c_3 \int_0^T \|v_m(t) - v_l(t)\|_{L^2(\mathbb{R}^n)}^2 dt + \varepsilon \tilde{K}^{m,l}(T), \end{aligned} \quad (2.12)$$

where

$$\tilde{K}^{m,l}(t) := \int_0^t \tau \left(f(\|\nabla v_m(\tau)\|_{L^2(\mathbb{R}^n)}) - f(\|\nabla v_l(\tau)\|_{L^2(\mathbb{R}^n)}) \right) \|\nabla v_l(\tau)\|_{L^2(\mathbb{R}^n)} \|v_m(\tau) - v_l(\tau)\|_{L^2(\mathbb{R}^n)} d\tau,$$

and by (1.4) and (2.7), it is easy to see that

$$\limsup_{m \rightarrow \infty} \limsup_{l \rightarrow \infty} \left\| \tilde{K}^{m,l} \right\|_{C[0,T]} = 0.$$

Adding (2.11) to (2.12) and choosing γ and ε small enough, we obtain

$$\begin{aligned} T E(v_m(T) - v_l(T)) & \leq c_4 \int_0^T t E(v_m(t) - v_l(t)) \|v_{mt}(t)\|_{L^2(\mathbb{R}^n)}^2 dt \\ & + c_4 \int_0^T E(v_m(t) - v_l(t)) dt + c_4 |K^{m,l}(T)| + c_4 |\tilde{K}^{m,l}(T)|, \quad \forall T \geq 0. \end{aligned}$$

Now, denoting $y_{m,l}(t) := t E(v_m(t) - v_l(t))$ and applying Gronwall inequality, we get

$$\begin{aligned} & y_{m,l}(T) \\ & \leq c_4 \left(\int_0^T E(v_m(t) - v_l(t)) dt + \|K^{m,l}\|_{C[0,T]} + \|\tilde{K}^{m,l}\|_{C[0,T]} \right) e^{\int_0^T \|v_{mt}(t)\|_{L^2(\mathbb{R}^n)}^2 dt}, \end{aligned}$$

which, together with (2.8), yields

$$T E(v_m(T) - v_l(T)) \leq c_5 \left(\int_0^T E(v_m(t) - v_l(t)) dt + \|K^{m,l}\|_{C[0,T]} + \|\tilde{K}^{m,l}\|_{C[0,T]} \right),$$

for every $T \geq 0$. Passing to the limit in the above inequality and taking into account (2.10), we find

$$\limsup_{m \rightarrow \infty} \limsup_{l \rightarrow \infty} T E(v_m(T) - v_l(T)) \leq c_6, \quad \forall T \geq 0,$$

which gives

$$\limsup_{m \rightarrow \infty} \limsup_{l \rightarrow \infty} \|S(t_{k_m})\varphi_{k_m} - S(t_{k_l})\varphi_{k_l}\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} \leq \frac{c_7}{\sqrt{T}}, \quad \forall T > 0.$$

Consequently, we have

$$\liminf_{l \rightarrow \infty} \liminf_{m \rightarrow \infty} \|S(t_k)\varphi_k - S(t_m)\varphi_m\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} = 0.$$

Thus, by using the argument at the end of the proof of [20, Lemma 3.4], we complete the proof of the lemma. \square

Since, by (1.3) and (1.5), problem (1.1)-(1.2) admits a strict Lyapunov function

$$\Phi(u(t)) = E(u(t)) + \frac{1}{2}F\left(\|\nabla u(t)\|_{L^2(\mathbb{R}^n)}^2\right) - \int_{\mathbb{R}^n} h(x)u(t, x) dx,$$

applying [18, Corollary 7.5.7], we have the following theorem.

Theorem 2.1. *Under conditions (1.3)-(1.6) the semigroup $\{S(t)\}_{t \geq 0}$ generated by the problem (1.1)-(1.2) possesses a global attractor \mathcal{A} in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and $\mathcal{A} = \mathcal{M}^u(\mathcal{N})$.*

3. REGULARITY OF THE GLOBAL ATTRACTOR

To prove the regularity of the global attractor, we start with the following lemma.

Lemma 3.1. *Assume that the conditions (1.3) and (1.4) hold and B is a bounded subset in $H^4(\mathbb{R}^n) \times H^2(\mathbb{R}^n)$. Then there exists a constant $C > 0$ such that*

$$\sup_{\varphi \in B} \|S(t)\varphi\|_{H^4(\mathbb{R}^n) \times H^2(\mathbb{R}^n)} \leq C$$

for all $t \geq 0$.

Proof. Let $(u_0, u_1) \in B$ and $S(t)(u_0, u_1) := (u(t), u_t(t))$. Then we have $u \in C([0, \infty); H^4(\mathbb{R}^n)) \cap C^1([0, \infty); H^2(\mathbb{R}^n)) \cap C^2([0, \infty); L^2(\mathbb{R}^n))$. Defining

$$v(t, x) := \frac{u(t + \tau, x) - u(t, x)}{\tau}, \quad \tau > 0,$$

by (1.1), we get

$$\begin{aligned} & v_{tt}(t, x) + \Delta^2 v(t, x) + \alpha(x)v_t(t, x) + \lambda v(t, x) - f\left(\|\nabla u(t)\|_{L^2(\mathbb{R}^n)}\right) \Delta v(t, x) \\ & - \frac{f(\|\nabla u(t + \tau)\|_{L^2(\mathbb{R}^n)}) - f(\|\nabla u(t)\|_{L^2(\mathbb{R}^n)})}{\tau} \Delta u(t + \tau, x) = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n. \end{aligned} \quad (3.1)$$

Multiplying (3.1) by v_t and integrating over \mathbb{R}^n , we find

$$\begin{aligned} & \frac{d}{dt}E(v(t)) + \int_{\mathbb{R}^n} \alpha(x)|v_t(t, x)|^2 dx + \frac{1}{2}f\left(\|\nabla u(t)\|_{L^2(\mathbb{R}^n)}\right) \frac{d}{dt}\left(\|\nabla v(t)\|_{L^2(\mathbb{R}^n)}^2\right) \\ & - \frac{f(\|\nabla u(t + \tau)\|_{L^2(\mathbb{R}^n)}) - f(\|\nabla u(t)\|_{L^2(\mathbb{R}^n)})}{\tau} \int_{\mathbb{R}^n} \Delta u(t + \tau, x)v_t(t, x) dx = 0. \end{aligned} \quad (3.2)$$

Since

$$\begin{aligned} & \left| \frac{d}{dt}f\left(\|\nabla u(t)\|_{L^2(\mathbb{R}^n)}\right) \right| = \left| \frac{f'\left(\|\nabla u(t)\|_{L^2(\mathbb{R}^n)}\right)}{\|\nabla u(t)\|_{L^2(\mathbb{R}^n)}} \langle \nabla u(t), \nabla u_t(t) \rangle_{L^2(\mathbb{R}^n)} \right| \\ & \leq \left| f'\left(\|\nabla u(t)\|_{L^2(\mathbb{R}^n)}\right) \right| \|\nabla u_t(t)\|_{L^2(\mathbb{R}^n)}, \quad \text{a.e. in } (0, \infty), \end{aligned}$$

considering (1.3) and (1.6) in (3.2), we obtain

$$\begin{aligned} & \frac{d}{dt} \left(E(v(t)) + \frac{1}{2} f \left(\|\nabla u(t)\|_{L^2(\mathbb{R}^n)} \right) \|\nabla v(t)\|_{L^2(\mathbb{R}^n)}^2 \right) + \alpha_0 \|v_t(t)\|_{L^2(\mathbb{R}^n)}^2 \\ & \leq c_1 \left(\|\nabla u_t(t)\|_{L^2(\mathbb{R}^n)} \|\nabla v(t)\|_{L^2(\mathbb{R}^n)}^2 + \|\nabla v(t)\|_{L^2(\mathbb{R}^n)} \int_{\mathbb{R}^n} \Delta u(t+\tau, x) v_t(t, x) dx \right) \\ & \leq c_2 \left(\|\nabla u_t(t)\|_{L^2(\mathbb{R}^n)} \|v(t)\|_{H^2(\mathbb{R}^n)} \|v(t)\|_{L^2(\mathbb{R}^n)} + \|v(t)\|_{H^2(\mathbb{R}^n)}^{\frac{1}{2}} \|v(t)\|_{L^2(\mathbb{R}^n)}^{\frac{1}{2}} \|v_t(t)\|_{L^2(\mathbb{R}^n)} \right). \end{aligned}$$

Since, by (1.6),

$$\|v(t)\|_{L^2(\mathbb{R}^n)} = \left\| \frac{u(t+\tau, x) - u(t, x)}{\tau} \right\|_{L^2(\mathbb{R}^n)} \leq \sup_{0 \leq t < \infty} \|u_t(t)\|_{L^2(\mathbb{R}^n)} < \widehat{C},$$

by the previous inequality, we get

$$\begin{aligned} & \frac{d}{dt} \left(E(v(t)) + \frac{1}{2} f \left(\|\nabla u(t)\|_{L^2(\mathbb{R}^n)} \right) \|\nabla v(t)\|_{L^2(\mathbb{R}^n)}^2 \right) + \alpha_0 \|v_t(t)\|_{L^2(\mathbb{R}^n)}^2 \\ & \leq c_3 \left(\|\nabla u_t(t)\|_{L^2(\mathbb{R}^n)} \|v(t)\|_{H^2(\mathbb{R}^n)} + \|v(t)\|_{H^2(\mathbb{R}^n)}^{\frac{1}{2}} \|v(t)\|_{L^2(\mathbb{R}^n)}^{\frac{1}{2}} \|v_t(t)\|_{L^2(\mathbb{R}^n)} \right). \end{aligned} \quad (3.3)$$

Multiplying (3.1) by εv and integrating over \mathbb{R}^n , we find

$$\begin{aligned} & \varepsilon \frac{d}{dt} \left(\langle \nabla v(t), \nabla v_t(t) \rangle_{L^2(\mathbb{R}^n)} + \frac{1}{2} \int_{\mathbb{R}^n} \alpha(x) v(t, x)^2 dx \right) + \varepsilon \|\Delta v(t)\|_{L^2(\mathbb{R}^n)}^2 \\ & + \varepsilon \lambda \|v(t)\|_{L^2(\mathbb{R}^n)}^2 \leq \varepsilon \|v_t(t)\|_{L^2(\mathbb{R}^n)}^2 + \varepsilon c_4 \|v(t)\|_{H^1(\mathbb{R}^n)}. \end{aligned} \quad (3.4)$$

Considering (3.3) and (3.4), for sufficiently small $\varepsilon > 0$ and applying Young inequality, we obtain

$$\frac{d}{dt} \Psi(t) + c_5 E(v(t)) \leq c_6 + c_6 \|\nabla u_t(t)\|_{L^2(\mathbb{R}^n)}^2, \quad (3.5)$$

where $c_5 > 0$ and

$$\Psi(t) := E(v(t)) + \frac{1}{2} f(\|\nabla u(t)\|) \|\nabla v(t)\|_{L^2(\mathbb{R}^n)}^2 + \varepsilon \langle \nabla v(t), \nabla v_t(t) \rangle_{L^2(\mathbb{R}^n)} + \frac{\varepsilon}{2} \int_{\mathbb{R}^n} \alpha(x) |v(t, x)|^2 dx.$$

Since $\varepsilon > 0$ is sufficiently small, there exist constants $c > 0$, $\tilde{c} > 0$ such that

$$cE(v(t)) \leq \Psi(t) \leq \tilde{c}E(v(t)). \quad (3.6)$$

Then, by (3.5) and (3.6), we have

$$\frac{d}{dt} \Psi(t) + c_7 \Psi(t) \leq c_6 + c_6 \|\nabla u_t(t)\|_{L^2(\mathbb{R}^n)}^2$$

which yields

$$\begin{aligned} \Psi(t) & \leq c_8 + e^{-c_7 t} c_8 \int_0^t \|\nabla u_t(s)\|_{L^2(\mathbb{R}^n)}^2 e^{c_7 s} ds \\ & \leq c_8 + c_8 \sup_{t \in [0, T]} \left(\|u_t(t)\|_{L^2(\mathbb{R}^n)} \|u_t(t)\|_{H^2(\mathbb{R}^n)} \right), \end{aligned}$$

for every $T \geq 0$. Taking into account (1.6) and (3.6) in the last inequality, we find

$$E(v(t)) \leq c_9 + c_9 \sup_{t \in [0, T]} \|u_t(t)\|_{H^2(\mathbb{R}^n)}, \quad \forall t \in [0, T],$$

for every $T \geq 0$. Passing to limit as $\tau \rightarrow 0$ in the above inequality, from the definition of v , we obtain

$$E(u_t(t)) \leq c_9 + c_9 \sup_{t \in [0, T]} \|u_t(t)\|_{H^2(\mathbb{R}^n)}, \quad \forall t \in [0, T],$$

for every $T \geq 0$. Thus, after taking supremum on $[0, T]$ and applying Young inequality, we have

$$E(u_t(t)) \leq c_{10}, \quad \forall t \geq 0.$$

Taking into account this estimate in (1.1), we find that

$$\|u(t)\|_{H^4(\mathbb{R}^n)} \leq c_{11}$$

which, together with previous inequality, yields

$$\|(u(t), u_t(t))\|_{H^4(\mathbb{R}^n) \times H^2(\mathbb{R}^n)} \leq C,$$

for some constant $C > 0$. □

Now we can show the regularity of the attractor.

Theorem 3.1. *Under the assumptions of Theorem 1.1, the global attractor \mathcal{A} for the problem (1.1)-(1.2) is bounded in $H^4(\mathbb{R}^n) \times H^2(\mathbb{R}^n)$.*

Proof. Let $\theta \in \mathcal{A}$. By the invariance of \mathcal{A} , it follows that (see [21, p. 159]) there exists an invariant trajectory $\gamma = \{U(t) = (u(t), u_t(t)) : t \in \mathbb{R}\} \subset \mathcal{A}$ such that $U(0) = \theta$. By an invariant trajectory we mean a curve $\gamma = \{U(t) : t \in \mathbb{R}\}$ such that $S(t)U(\tau) = U(t + \tau)$ for all $t \geq 0$ and $\tau \in \mathbb{R}$ (see [21, p. 157])

In the case when $h \equiv 0$ in equation (1.1), by (1.4), it follows that the stationary point set $\mathcal{N} = \{(0, 0)\}$. By Theorem 1.1 and the definition of unstable manifold, we have

$$\lim_{t \rightarrow -\infty} \inf_{w \in \mathcal{N}} \|U(t) - w\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} = 0. \quad (3.7)$$

Then, from the monotonicity of the Lyapunov function $\Phi(\cdot)$, we have $\mathcal{A} = \{(0, 0)\}$.

So, we will consider the case $h \neq 0$. In this case, it is clear that \mathcal{N} does not contain $(0, 0)$. Since \mathcal{N} is compact (because it is a closed subset of \mathcal{A}), by (3.7), we obtain that there exists $t_0 \in (-\infty, 0)$ such that

$$\|U(t)\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} > c_0, \quad \forall t \leq t_0, \quad (3.8)$$

for some $c_0 > 0$ which depends on \mathcal{N} and is independent of $U(t)$. Now, again defining

$$v(t, x) := \frac{u(t + \tau, x) - u(t, x)}{\tau},$$

we have equation (3.1). Multiplying (3.1) by v_t and integrating over \mathbb{R}^n , by using (3.8), we get

$$\begin{aligned} & \frac{d}{dt} \left(E(v(t)) + \frac{1}{2} f(\|\nabla u(t)\|) \|\nabla v(t)\|_{L^2(\mathbb{R}^n)}^2 \right) + \alpha_0 \|v_t(t)\|_{L^2(\mathbb{R}^n)}^2 \\ & \leq \frac{f'(\|\nabla u(t)\|)}{\|\nabla u(t)\|} \|\Delta u(t)\| \|u_t(t)\| \|\nabla v(t)\|_{L^2(\mathbb{R}^n)}^2 \end{aligned}$$

$$\begin{aligned}
& + \widehat{c}_1 \|\nabla v(t)\|_{L^2(\mathbb{R}^n)} \int_{\mathbb{R}^n} \Delta u(t+\tau, x) v_t(t, x) dx \\
& \leq \widehat{c}_2 \|\nabla v(t)\|_{L^2(\mathbb{R}^n)}^2 + \widehat{c}_2 \|\nabla v(t)\|_{L^2(\mathbb{R}^n)} \|v_t(t)\|_{L^2(\mathbb{R}^n)} \\
& \leq \widehat{c}_3 \|v(t)\|_{H^2(\mathbb{R}^n)} + \widehat{c}_3 \|v(t)\|_{H^2(\mathbb{R}^n)}^{\frac{1}{2}} \|v_t(t)\|_{L^2(\mathbb{R}^n)}.
\end{aligned} \tag{3.9}$$

By using (3.4), (3.6) and (3.9), we find

$$\frac{d}{dt} \Psi(t) + \widehat{c}_4 \Psi(t) \leq \widehat{c}_5, \quad \forall t \leq t_0,$$

which yields

$$\Psi(t) \leq \widehat{c}_6 + e^{\widehat{c}_4(s-t)} \Psi(s), \quad s \leq t \leq t_0,$$

where $\widehat{c}_4 > 0$. Then, passing to limit as $s \rightarrow -\infty$ and taking into account that $\cup_{t \in \mathbb{R}} U(t) \subset \mathcal{A}$, we have

$$\Psi(t) \leq \widehat{c}_6,$$

which, by (3.6), gives

$$E(v(t)) \leq \widehat{c}_7.$$

Now, passing to limit as $\tau \rightarrow 0$ in the last inequality, we obtain

$$E(u_t(t)) \leq \widehat{c}_7, \quad \forall t \leq t_0. \tag{3.10}$$

Considering (3.10) in (1.1), we find

$$\|u(t)\|_{H^4(\mathbb{R}^n)} \leq \widehat{c}_8, \quad \forall t \leq t_0,$$

which, together with (3.10), yields

$$\|(u(t), u_t(t))\|_{H^4(\mathbb{R}^n) \times H^2(\mathbb{R}^n)} \leq \widehat{c}_9, \quad \forall t \leq t_0.$$

Thus, applying Lemma 3.1 to the set $B = \{(u(t), u_t(t)) : t \in (-\infty, t_0]\}$, we obtain

$$\|\theta\|_{H^4(\mathbb{R}^n) \times H^2(\mathbb{R}^n)} \leq C,$$

where $C > 0$ is a constant independent of θ . □

4. FINITE DIMENSIONALITY OF THE GLOBAL ATTRACTOR

In this section, we will use the idea of the [12] to obtain the finite dimensionality. Let us start with the following lemma.

Lemma 4.1. *Assume that the conditions (1.3) and (1.4) hold and $u \in W^{1,\infty}(0, \infty; H^2(\mathbb{R}^n))$ such that*

$$\|u\|_{W^{1,\infty}(0, \infty; H^2(\mathbb{R}^n))} + \int_0^\infty \|u_t(t)\|_{L^2(\mathbb{R}^n)}^2 dt < c. \tag{4.1}$$

for some constant $c > 0$. Also, let $\{T(t, \tau)\}_{t \geq \tau}$ be the process generated by the problem

$$\begin{cases} v_{tt} + \Delta^2 v + \alpha(x)v_t + \lambda v - f(\|\nabla u(t)\|_{L^2(\mathbb{R}^n)}) \Delta v = 0, & t \geq \tau, \\ v(\tau) = v_0, \quad v_t(\tau) = v_1, & \tau \geq 0 \end{cases} \tag{4.2}$$

in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. Then there exist $M = M(c) > 1$ and $\omega = \omega(c) > 0$ such that

$$\|T(t, \tau)\|_{L(H^{2(1+i)}(\mathbb{R}^n) \times H^{2i}(\mathbb{R}^n))} \leq M e^{-\omega(t-\tau)}, \quad \forall t \geq \tau$$

where $i = 0, 1$ and $L(X)$ is the space of linear bounded operators in X .

Proof. By using the multiplier $(v_t + \varepsilon v)$ as in Lemma 3.1, for sufficiently small $\varepsilon > 0$ and applying Young inequality, we get

$$\frac{d}{dt} \Psi(t) + \gamma E(v(t)) \leq c_1 \|\nabla u_t(t)\|_{L^2(\mathbb{R}^n)} \|\nabla v(t)\|_{L^2(\mathbb{R}^n)}^2,$$

for some $\gamma > 0$. Then, by using interpolation, (3.6) and (4.1), we find

$$\frac{d}{dt} \Psi(t) + \gamma \Psi(v(t)) \leq c_2 \|u_t(t)\|_{L^2(\mathbb{R}^n)}^{\frac{1}{2}} \Psi(t).$$

Hence, by Gronwall inequality and (3.6), we have,

$$E(t) \leq c_3 E(\tau) e^{\frac{c_2}{\tau} \int_{\tau}^t \|u_t(\sigma)\|_{L^2(\mathbb{R}^n)}^{\frac{1}{2}} d\sigma - \gamma(t-\tau)}. \quad (4.3)$$

Since, by Holder inequality and (4.1),

$$\int_{\tau}^t \|u_t(\sigma)\|_{L^2(\mathbb{R}^n)}^{\frac{1}{2}} d\sigma \leq c_4 (t-\tau)^{\frac{3}{4}},$$

from (4.3) it follows that

$$\|T(t, \tau)\|_{L(H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n))} \leq M_1 e^{-\omega(t-\tau)}, \quad \forall t \geq \tau, \quad (4.4)$$

for some $M_1 > 1$ and $\omega > 0$.

Now, let us define $w := v_t$. Then w is the solution of the following equation

$$\begin{aligned} w_{tt}(t, x) + \Delta^2 w(t, x) + \alpha(x) w_t(t, x) + \lambda w(t, x) - f\left(\|\nabla u(t)\|_{L^2(\mathbb{R}^n)}\right) \Delta w(t, x) \\ - \frac{f'\left(\|\nabla u(t)\|_{L^2(\mathbb{R}^n)}\right)}{\|\nabla u(t)\|_{L^2(\mathbb{R}^n)}} \langle \nabla u_t(t), \nabla u(t) \rangle_{L^2(\mathbb{R}^n)} \Delta v(t, x) = 0, \quad t \geq \tau, x \in \mathbb{R}^n, \end{aligned}$$

and by the variation of parameters formula, we have

$$W(t) = T(t, \tau) W(\tau) + \int_{\tau}^t T(t, s) G(s) ds,$$

where $W(t) := (w(t), w_t(t))$ and $G(t) := \left(0, \frac{f'(\|\nabla u(t)\|_{L^2(\mathbb{R}^n)})}{\|\nabla u(t)\|_{L^2(\mathbb{R}^n)}} \langle \nabla u_t(t), \nabla u(t) \rangle_{L^2(\mathbb{R}^n)} \Delta v(t)\right)$. Therefore, by using (4.4), we find

$$\begin{aligned} \|W(t)\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} &\leq \|T(t, \tau) W(\tau)\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} + \int_{\tau}^t \|T(t, s) G(s)\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} ds \\ &\leq M_1 e^{-\omega(t-\tau)} \|W(\tau)\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} + c_5 \int_{\tau}^t e^{-\omega(t-s)} \|G(s)\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} ds \end{aligned}$$

$$\begin{aligned}
&\leq M_1 e^{-\omega(t-\tau)} \|W(\tau)\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} + c_6 \int_{\tau}^t e^{-\omega(t-s)} \|v(s)\|_{H^2(\mathbb{R}^n)} ds \\
&\leq M_1 e^{-\omega(t-\tau)} \|W(\tau)\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} + c_7 \int_{\tau}^t e^{-\omega(t-s)} e^{-\omega(s-\tau)} \|(v(\tau), v_t(\tau))\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} ds \\
&\leq c_8 e^{-\omega(t-\tau)} \left(\|W(\tau)\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} + \|(v(\tau), v_t(\tau))\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} \right), \quad \forall t \geq \tau.
\end{aligned}$$

Thus, the last inequality, together with (4.2)₁, gives

$$\|T(t, \tau)\|_{L(H^4(\mathbb{R}^n) \times H^2(\mathbb{R}^n))} \leq M_2 e^{-\omega(t-\tau)}, \quad \forall t \geq \tau,$$

for some $M_2 > 1$. □

Now, we can give the theorem about the finite dimensionality of the global attractor.

Theorem 4.1. *The fractal dimension of the global attractor \mathcal{A} is finite.*

Proof. Let $\theta_1, \theta_2 \in \mathcal{A}$ and $(u(t), u_t(t)) = S(t)\theta_1$, $(v(t), v_t(t)) = S(t)\theta_2$. Define $w(t) := v(t) - u(t)$. Then, we find

$$\begin{aligned}
&w_{tt}(t, x) + \Delta^2 w(t, x) + \alpha(x)w_t(t, x) + \lambda w(t, x) - f\left(\|\nabla u(t)\|_{L^2(\mathbb{R}^n)}\right) \Delta w(t, x) \\
&\quad - \left(f\left(\|\nabla v(t)\|_{L^2(\mathbb{R}^n)}\right) - f\left(\|\nabla u(t)\|_{L^2(\mathbb{R}^n)}\right)\right) \Delta v(t, x) = 0.
\end{aligned} \tag{4.5}$$

Hence, by the variation of parameters formula, we have

$$(w(t), w_t(t)) = T(t, 0)(w(0), w_t(0)) + \int_0^t T(t, \tau) \widehat{G}(\tau) d\tau,$$

where $\widehat{G}(t) = \left(0, \left(f\left(\|\nabla v(t)\|_{L^2(\mathbb{R}^n)}\right) - f\left(\|\nabla u(t)\|_{L^2(\mathbb{R}^n)}\right)\right) \Delta v(t)\right)$. By Lemma 4.1, we get

$$\begin{aligned}
&\|S(t)\theta_2 - S(t)\theta_1\|_{H^4(\mathbb{R}^n) \times H^2(\mathbb{R}^n)} \\
&\leq M e^{-\omega t} \|\theta_2 - \theta_1\|_{H^4(\mathbb{R}^n) \times H^2(\mathbb{R}^n)} + \widetilde{c}_1 \int_0^t e^{-\omega(t-\tau)} \|S(\tau)\theta_2 - S(\tau)\theta_1\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} d\tau.
\end{aligned} \tag{4.6}$$

Applying Gronwall lemma to (4.6), we obtain

$$\|S(t)\theta_2 - S(t)\theta_1\|_{H^4(\mathbb{R}^n) \times H^2(\mathbb{R}^n)} \leq M e^{(\widetilde{c}_2 - \omega)t} \|\theta_2 - \theta_1\|_{H^4(\mathbb{R}^n) \times H^2(\mathbb{R}^n)}, \quad \forall t \geq 0. \tag{4.7}$$

Also, by (4.6), we have

$$\begin{aligned}
&\|S(t)\theta_2 - S(t)\theta_1\|_{H^4(\mathbb{R}^n) \times H^2(\mathbb{R}^n)} \\
&\leq M e^{-\omega t} \|\theta_2 - \theta_1\|_{H^4(\mathbb{R}^n) \times H^2(\mathbb{R}^n)} + \frac{\widetilde{c}_1}{\omega} \sup_{0 \leq \tau \leq t} \|S(\tau)\theta_2 - S(\tau)\theta_1\|_{H^2(B(0,r)) \times L^2(B(0,r))} \\
&\quad + \widetilde{c}_1 \int_0^t e^{-\omega(t-\tau)} \|S(\tau)\theta_2 - S(\tau)\theta_1\|_{H^2(\mathbb{R}^n \setminus B(0,r)) \times L^2(\mathbb{R}^n \setminus B(0,r))} d\tau, \quad \forall t \geq 0 \text{ and } \forall r > 0,
\end{aligned} \tag{4.8}$$

where $B(0, r) = \{x : x \in \mathbb{R}^n, |x| < r\}$.

Now, we will estimate the integral term on the right hand side of (4.8). Let $\eta \in C^\infty(\mathbb{R}^n)$, $0 \leq \eta(x) \leq 1$, $\eta(x) = \begin{cases} 0, & |x| \leq 1 \\ 1, & |x| \geq 2 \end{cases}$ and $\eta_r(x) = \eta(\frac{x}{r})$. Multiplying (4.5) by η_r and denoting $w_r(t) = \eta_r w(t)$, we get

$$\begin{aligned} w_{rtt}(t, x) + \Delta^2 w_r(t, x) + \alpha(x) w_{rt}(t, x) + \lambda w_r(t, x) - f\left(\|\nabla u(t)\|_{L^2(\mathbb{R}^n)}\right) \Delta w_r(t, x) \\ - \eta_r \left(f\left(\|\nabla v(t)\|_{L^2(\mathbb{R}^n)}\right) - f\left(\|\nabla u(t)\|_{L^2(\mathbb{R}^n)}\right) \right) \Delta v(t, x) = f_1(t), \end{aligned}$$

where

$$\begin{aligned} f_1(t) = \Delta^2 \eta_r w + 2\Delta \eta_r \Delta w + 2 \sum_{i=1}^n (\Delta \eta_r)_{x_i} w_{x_i} + 2 \sum_{i=1}^n (\eta_r)_{x_i} \Delta w_{x_i} + 4 \sum_{i,j=1}^n (\eta_r)_{x_i x_j} w_{x_i x_j} \\ - \Delta \eta_r f\left(\|\nabla u(t)\|_{L^2(\mathbb{R}^n)}\right) w - 2f\left(\|\nabla u(t)\|_{L^2(\mathbb{R}^n)}\right) \sum_{i=1}^n (\eta_r)_{x_i} w_{x_i}. \end{aligned}$$

Then, by the variation of parameters formula, we have

$$(w_r(t), w_{rt}(t)) = T(t, 0)(w_r(0), w_{rt}(0)) + \int_0^t T(t, \tau) G_r(\tau) d\tau, \quad (4.9)$$

where

$$G_r(t) := \left(0, \eta_r \left(f\left(\|\nabla v(t)\|_{L^2(\mathbb{R}^n)}\right) - f\left(\|\nabla u(t)\|_{L^2(\mathbb{R}^n)}\right) \right) \Delta v(t) + f_1(t) \right).$$

Hence, applying Lemma 4.1 to (4.9) and taking into account (4.7), we obtain

$$\begin{aligned} \|(w_r(t), w_{rt}(t))\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} &\leq M e^{-\omega t} \|\theta_2 - \theta_1\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} \\ &+ \tilde{c}_3 e^{-\omega t} \int_0^t e^{\omega \tau} \|\Delta v(\tau)\|_{L^2(\mathbb{R}^n \setminus B(0, r))} \|w(\tau)\|_{H^1(\mathbb{R}^n)} d\tau + \frac{\tilde{c}_3}{r} e^{-\omega t} \int_0^t e^{\omega \tau} \|w(\tau)\|_{H^4(\mathbb{R}^n)} d\tau \\ &\leq \tilde{c}_4 \left(e^{-\omega t} + \Pi_r e^{(\tilde{c}_1 - \omega)t} \right) \|\theta_2 - \theta_1\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)}, \quad \forall t \geq 0 \text{ and } \forall r \geq 1, \end{aligned}$$

where

$$\Pi_r := \sup_{t \geq 0} \|\Delta v(t)\|_{L^2(\mathbb{R}^n \setminus B(0, r))} + \frac{1}{r}.$$

Then, the last inequality, together with (4.7), gives

$$\begin{aligned} \int_0^t e^{-\omega(t-\tau)} \|S(\tau) \theta_2 - S(\tau) \theta_1\|_{H^2(\mathbb{R}^n \setminus B(0, r)) \times L^2(\mathbb{R}^n \setminus B(0, r))} d\tau \\ \leq \tilde{c}_5 \left(e^{-\omega t} + \Pi_r e^{(\tilde{c}_2 - \omega)t} \right) t \|\theta_2 - \theta_1\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)}, \quad \forall t \geq 0 \text{ and } \forall r \geq 1. \end{aligned} \quad (4.10)$$

So, by (4.8) and (4.10), we have

$$\begin{aligned} \|S(t) \theta_2 - S(t) \theta_1\|_{H^4(\mathbb{R}^n) \times H^2(\mathbb{R}^n)} &\leq \tilde{c}_6 \left(e^{-\omega t} + e^{-\omega t} t + \Pi_r e^{(\tilde{c}_2 - \omega)t} t \right) \|\theta_2 - \theta_1\|_{H^4(\mathbb{R}^n) \times H^2(\mathbb{R}^n)} \\ &+ \tilde{c}_6 \sup_{0 \leq \tau \leq t} \|S(\tau) \theta_2 - S(\tau) \theta_1\|_{H^2(B(0, r)) \times L^2(B(0, r))}, \quad \forall t \geq 0 \text{ and } \forall r \geq 1. \end{aligned} \quad (4.11)$$

From the compactness of \mathcal{A} , it follows that $\Pi_r \rightarrow 0$, uniformly with respect to the trajectories from \mathcal{A} , as $r \rightarrow \infty$. Thus, applying [18, Theorem 7.9.6], by (4.7) and (4.11), we obtain the finite dimensionality of \mathcal{A} . \square

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